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An eigenvalue study of a viscous flow separated by mass addition

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SUMMARY
Eigenvalue problem methods, developed for boundary-layer flow are used to consider the spatial stability of the viscous flow past a flat plate which has been separated by mass addition at the surface. A study is made of the rate of approach of a slightly disturbed initial profile to the interaction similarity solution found by Kassoy [1] and Klemp and Acrivos [2]. It is shown that eigenfunctions generated in the separated viscous layer (free shear layer) propagate into the inviscid rotational layer adjacent to the wall. Thus by the inherent interaction process involved, these disturbances affect the external flow as well. The results indicate a relatively slow rate of decay when compared to an attached boundary-layer flow on an impermeable surface.

1. Introduction

Recently Kassoy [1] and Klemp and Acrivos [2] developed an interaction theory used to describe separated viscous flow past a flat plate with surface mass addition. A calculation was carried out in detail for the similarity injection distribution $v_w(x, 0) = C/(2\,\mathrm{Re}\,x)^{\frac{1}{2}}$ where C is greater than the critical blowoff value $C_0 = 0.87574\ldots$, found from boundary-layer theory [1]. Here $(\mathrm{Re}\,x) \equiv U'_\infty x'/v'_\infty$, the local Reynolds number defined in terms of the reference velocity, kinematic viscosity and the dimensional streamwise variable. The flow structure consists first of a wall layer of $O(\mathrm{Re}^{-\frac{1}{3}})$ in extent, composed of inviscid, rotational injectant fluid. Above this lies a viscous free shear layer of thickness $O(\mathrm{Re}^{-\frac{1}{2}})$. These two, relatively thin, internal layers act as an effective displacement body which disturbs the basically uniform inviscid, irrotational external flow to $O(\mathrm{Re}^{-\frac{1}{3}})$. The resulting interaction, described by slender body theory, produces an $O(\mathrm{Re}^{-\frac{1}{3}})$ favorable pressure gradient which is essential (from the physical point of view) for injection rates measured by $C > C_0$.

In the present work eigenvalue procedures, described by Libby [3], Stewartson [4], Libby and Chen [5], Kemp [6] for boundary-layer flows, are adopted for a study of the spatial stability of the above interaction problem. Interest is focused on the spatial decay of an initial velocity profile (at a point x_i) toward the similarity solution described in [1] in the limit $x \to \infty$.

In the usual boundary-layer eigenvalue problem, one is concerned primarily with the nature of the eigenfunctions in the viscous layer alone. Presumably their influence on the displacement effect of the boundary layer could be calculated in terms of an $O(\mathrm{Re}^{-\frac{1}{2}})$ correction to the external flow. However, this does not appear to have been considered. In the present problem this displacement interaction effect is, of course, absolutely essential. The eigenfunctions here are generated in the free shear layer. As such they are the disturbances associated with the similarity form of the free shear layer, Lock's mixing layer solution [1]. Formal matching conditions between the shear and wall layers can be used to show that the very same eigenfunction disturbances must appear in the latter layer as well. This inviscid, rotational layer exhibits only the shear-layer induced eigenfunctions because it does not generate any of its own. The presence of these disturbances in the wall layer causes the basic shape of the effective displacement body to be altered to $O(\mathrm{Re}^{-\frac{1}{3}})$. Hence the disturbances alter the external uniform stream to $O(\mathrm{Re}^{-\frac{1}{3}})$.

It may be observed then that the multilayer structure of the flow and the interaction nature of the problem lead to phenomena which are not observed in a boundary-layer calculation.

The eigenvalue computation for the free shear layer is of interest itself since the eigenvalues for Lock's mixing layer similarity solution have not been obtained previously. The numerical calculation involves an adaptation of the methods described by Libby and Chen [5] to a doubly infinite field. In addition large eigenvalues have been calculated analytically by means of asymptotic techniques described by Kemp [6]. Here again, these procedures must be generalized due to the field size.

2. Basic mathematical system

The mathematical system which describes the problem can be written in the form

$$\left[\psi_y \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial y} - \mathrm{Re}^{-1}\nabla^2\right]\nabla^2\psi = 0 \tag{1a}$$

$$\psi_y\ (\mathrm{r}\to\infty) = 1 \tag{1b}$$

$$\psi_y(x, 0) = 0, \quad x > 0 \tag{1c}$$

$$\psi_x(x, 0) = -C(2x\,\mathrm{Re})^{-\frac{1}{2}}, \quad x > 0,\ C > C_0 = 0.87574\ldots \tag{1d}$$

$$\psi_y(x_i, y) = \mathrm{Re}^{-\frac{1}{6}}\bar{u}_i(\bar{y}), \qquad 0 \leqq \bar{y} \leqq \bar{y}^*(x),\ x_i > 0 \tag{1e}$$

$$\psi_y(x_i, y) = \tilde{u}_i(z) \qquad -\infty < z < \infty,\ x_i > 0\,. \tag{1f}$$

The variables ψ, x and y in (1) are defined in terms of the uniform external flow U'_∞ and a length scale L'. The Reynolds number Re is defined in the usual way. In (1e, f) the nonsimilar initial profiles in the wall and shear layers are represented formally. Here $\bar{u}_i(\bar{y})$ and $\tilde{u}_i(z)$ represent "initial" values for the velocity profile at x_i in the wall layer and shear layer respectively. The variable $\bar{y} = y\,\mathrm{Re}^{\frac{1}{3}}$, and $\bar{y}^*(x)$ represents the outer edge of the injectant layer. The shear layer variable $z = [y - y_0(x, \mathrm{Re})]\,\mathrm{Re}^{\frac{1}{2}}$, where $y_0(x, \mathrm{Re})$ represents the location of the zero streamline. These initial profiles are to be considered as slightly perturbed from the relevant similarity solutions. It is to be determined whether these slightly altered profiles are able to relax to the appropriate similarity profiles in the limit of $x \to \infty$.

Unlike all previous calculations of this type of spatial stability where only the boundary-layer equations were considered, this problem requires a development in terms of the three distinct layers in this problem. Simply put, the disturbances in the shear layer alter the nature of the wall layer. And through the interaction coupling of the wall and external flows, a further disturbance to the latter appears. Hence we must develop slightly nonsimilar solutions in each of innermost layers and an appropriately altered solution in the external flow.

3. Formulation

The wall-layer transformations [1]

$$\mathrm{Re}^{\frac{1}{2}}\psi = \bar{\psi} = (2x)^{\frac{1}{2}} f(x, \eta), \quad \eta = \bar{y}\,(2x)^{-\frac{2}{3}},\ \bar{p} = p\mathrm{Re}^{\frac{1}{3}} \tag{2}$$
$$\bar{p}'(x) = -(2x)^{-\frac{4}{3}}\beta(x)$$

are substituted into (1) and the limit $\mathrm{Re}\to\infty$ applied. The lowest order system has the form

$$f f_{\eta\eta} + (\tfrac{1}{3}) f_\eta^2 = -\beta(x) + 2x(f_\eta f_{\eta x} - f_x f_{\eta\eta}) \tag{3a}$$

$$f_\eta(x, 0) = 0,\ f(x, 0) = -C \qquad x > x_i \tag{3b}$$

$$f_\eta(x_i, \eta) = (2x_i)^{-\frac{1}{6}}\bar{u}_i(\bar{y})\,. \tag{3c}$$

Here $\beta(x)$ is the initially unknown reduced interaction pressure gradient. The solution to (3) is sought in terms of perturbations from the asymptotically valid similarity solution described by Kassoy [1] or Klemp and Acrivos [2]. Hence f and β can be constructed in terms of expansions valid in the limit $x \to \infty$;

$$f(x,\eta)=f_0(\eta)+\sum_{n=1} f_n(x,\eta),\ \lim_{x\to\infty}\frac{f_{n+1}}{f_n}=0 \tag{4a}$$

$$\beta(x)=\beta_0+\sum_{n=1}\beta_n(x),\qquad \lim_{x\to\infty}\frac{\beta_{n+1}}{\beta_n}=0 \tag{4b}$$

The lowest order terms in (4) are described by

$$\eta=-(3\beta_0)^{-\frac{1}{2}}\int_c^{-f_0}[(C/\sigma)^{\frac{2}{3}}-1]^{-\frac{1}{2}}\,d\sigma \tag{5a}$$

$$\beta_0=(8/9(3^{\frac{1}{2}}))\eta^* \tag{5b}$$

$$f_0(\eta^*)=-C_0=-0.87574\ldots \tag{5c}$$

Eq. (5c), which effectively defines η^*, is derived from the matching condition between the wall and shear layers [1].

The corrections to $f_0(\eta)$, β_0 in (4) will be shown to result from disturbance phenomena occurring in the shear layer. The system describing the latter can be developed by using the shear layer transformations

$$\mathrm{Re}^{\frac{1}{2}}\psi=\tilde{\psi}=(2x)^{\frac{1}{2}}F(x,s),\ s=z(2x)^{-\frac{1}{2}}=[y-y_0(x,\mathrm{Re})](\mathrm{Re}/2x)^{\frac{1}{2}},\ \bar{p}=p\,\mathrm{Re}^{\frac{1}{3}} \tag{6}$$

in (1). It is found that in the limit $\mathrm{Re}\to\infty$

$$F_{sss}+FF_{ss}=2x(F_sF_{sx}-F_{ss}F_x) \tag{7a}$$

$$F(x,s=0)=0 \tag{7b}$$

$$F_s(x_i,s)=\tilde{u}_i(z) \tag{7c}$$

The pressure term does not appear in (7a) because it is asymptotically small in the limit $\mathrm{Re}\to\infty$ with respect to the remaining viscous and inertia terms.

The matching condition with the external flow implies that

$$F_s(x,s\to\infty)=1\,. \tag{7d}$$

Similarly the velocity match with the wall layer solution, $\tilde{\psi}_z(x,z\to-\infty)\sim\mathrm{Re}^{-\frac{1}{6}}\bar{\psi}_y(x,\bar{y}\to\bar{y}^*)$ where $\bar{y}^*=(2x)^{\frac{2}{3}}\eta^*$, indicates that

$$F_s(x,s\to-\infty)=0\,. \tag{7e}$$

Solutions for (7) valid in the limit $x\to\infty$ are described by

$$F(x,s)=F_0(s)+\sum_{n=1}F_n(x,s)\,,\ \lim_{x\to\infty}\frac{F_{n+1}}{F_n}=0\,. \tag{8}$$

The lowest order solution, $F_0(s)$, is simply Lock's free shear layer which is described by

$$F_0'''(s)+F_0F_0''=0$$
$$F_0'(s\to\infty)=1,\quad F_0(0)=0,\quad F_0'(s\to-\infty)=0\,.$$

It follows that $F_0(s\to-\infty)=-C_0+O(e^{C_0s})$. Hence streamfunction matching between the wall and shear layers leads to (5c).

The quantity η^* and (2) can be used to define a line, $y^*=\mathrm{Re}^{-\frac{1}{3}}(2x)^{\frac{2}{3}}\eta^*$, representing the shape of the injectant region to a first approximation. Since the shear layer is relatively thinner $[O(\mathrm{Re}^{-\frac{1}{2}})]$, this line also represents the shape of the effective displacement body for purposes of calculating corrections to the external flow. This can be developed formally by a higher order streamfunction matching between the shear layer and the external flow. To this end the latter can be written as

$$\psi\sim y+\mathrm{Re}^{-\frac{1}{3}}\psi_1(x,y)+\ldots$$

where the correction streamfunction is described by $\nabla^2\psi_1=0$, $\psi_1(r\to\infty)=0$. The required matching condition is constructed from (6) and the definition

$$y_0(x, \mathrm{Re}) \sim \mathrm{Re}^{-\frac{1}{2}}[(2x)^{\frac{2}{3}}\eta^* + \sum_{n=1} g_n(x)] + O(\mathrm{Re}^{-\frac{1}{2}}) \tag{9}$$

where $\lim_{x\to\infty}(g_{n+1}/g_n)=0$. In (9) the first approximation to the zero streamline location is identical to y^* because the shear-layer is thin compared to $O(\mathrm{Re}^{-\frac{1}{2}})$. The functions $g_n(x)$ represent corrections to the leading similarity result due to the nonsimilar initial data. It follows that the matching condition is

$$\psi_1(x, 0) = -[(2x)^{\frac{2}{3}}\eta^* + \sum_{n=1} g_n(x)]\,. \tag{10}$$

Hence the interaction pressure field can be calculated from the classical incompressible slender body formula

$$\bar{p} = -\frac{1}{\pi}\int_0^\infty \frac{\psi_{1\xi}(\xi, 0)d\xi}{(x-\xi)}\,. \tag{11}$$

4. Eigenvalue problem

The first correction to $F_0(s)$ is described by the $F_1(x, s)$ system. This is derived by substituting (8) into (7) and gathering appropriate terms. Thus

$$F_{1sss}+F_0F_{1ss}+F_0''F_1-2x[F_0'F_{1xs}-F_0''F_{1x}]=0 \tag{12a}$$

$$F_{1s}(x, s\to\infty)=F_1(x, 0)=F_{1s}(x, s\to-\infty)=0\,. \tag{12b}$$

Here primes refer to derivatives with respect to s. Eq. (12) describes the eigenvalue problem for Lock's free shear layer. The usual separation of variables procedure [3] indicates that

$$F_1(x, s) = \sum_{n=1}^{\infty} a_n(x/x_i)^{-\lambda_n/2}N_n(s) \tag{13}$$

where the a_n are Fourier constants which can be calculated in the usual manner. Here $N_n(s)$ is described by

$$N_n'''+F_0N_n''+\lambda_nF_0'N_n'+(1-\lambda_n)F_0''N_n=0 \tag{14a}$$

$$N_n'(\infty)=N_n(0)=N_n'(-\infty)=0\,. \tag{14b}$$

Stewartson [4] showed that F_0' is an eigenfunction for all λ_n and that there are two exact solutions.

$$\lambda_1=1 \quad N_1=F_0'-F_0'(0) \tag{15a}$$

$$\lambda_2=2 \quad N_2=F_0-\eta F_0'\,. \tag{15b}$$

The transformations $N_n(s)=M_n(s)F_0'$, $H_n=M_n'$ can be used in (14) to produce a second order Sturm–Liouville system for H_n [3]. The operator is self-adjoint. The eigenvalues are real and positive and the eigenfunctions orthogonal in $(-\infty, \infty)$ with respect to the weighting function $(F_0'^4/F_0'')$ (see Appendix A). The positiveness of the eigenvalues is a measure of the spatial stability implied by $F_1(x\to\infty, s)\to 0$. Hence the Fourier coefficients are described by

$$a_n = C_n^{-1}\int_{-\infty}^{+\infty}(F_0'^4/F_0'')(N_n/F_0')'[(F(x_i, s)-F_0)/F_0']'\,ds$$

where the square of the norm, C_n, is given by

$$\int_{-\infty}^{+\infty}(F_0'^4/F_0'')(N_n/F_0')'(N_m/F_0')'\,ds=\delta_{mn}C_n\,.$$

4.1. *The numerical solution for the eigenvalues*

The numerical procedure used to compute the eigenvalues of (14) is adopted from the quasilinearization method of Libby and Chen [5]. However several alterations are required to accommodate the doubly infinite field. For numerical purposes the last boundary condition in (14b) is replaced by an analytical equivalent derived from (14a) for $s \to -\infty$. Here F_0 is replaced by its asymptotic form, $-C_0 + O(e^{C_0 s})$. Then in the limit $s \to -\infty$, the equation has the form $N_n''' - C_0 N_n'' = O(e^{C_0 s})$. An integration and application of the last condition in (14b) produces the result

$$\lim_{s \to -\infty} [N_n'' \sim C_0 N_n'] \tag{16}$$

The first condition in (14b) is replaced by a formal statement of exponential decay. This is obtained by using the asymptotic form of Lock's solution,

$$F_0(s \to \infty) \sim s - k + \gamma (s-k)^{-2} \exp[-(\tfrac{1}{2})(s-k)^2]$$

$$k = 0.3739, \quad \gamma = 0.198$$

in (14a). It follows that

$$N_n''' + (s-k) N_n'' + \lambda_n N_n' \simeq \gamma N_n(\infty)(\lambda_n - 1) \exp[-(\tfrac{1}{2})(s-k)^2] \,. \tag{17}$$

for $s \to \infty$. The asymptotic form of F_0 can be used to show that this result is valid for $s \gtrsim 4$. An integration and application of the first condition in (14b) provides a relation between N_n' and N_n'' in the limit $s \to \infty$

$$\lim_{s \to \infty} \{N_n'' \simeq -(s-k)(1 + (1-\lambda_n)(s-k)^{-2}) N_n' + (1-\lambda_n)\gamma N_n(\infty)(s-k)^{-1} \exp[-(\tfrac{1}{2})(s-k)^2]\} \,. \tag{18}$$

The asymptotic expressions used to produce (18) can be used to show that the result is accurate for $s \geq s^*$ where s^* is sufficiently large so that $|(1-\lambda_n)/(s^* - k)^2| \ll 1$. Finally all the eigenfunctions except that corresponding to $\lambda = 2$ are normalized by $N_n'(0) = 1$. In the exceptional case (see (15b)), the normalization $N_n''(0) = 1$ is used.

The numerical procedure now parallels Libby and Chen (1968). Eqs. (14a) and (17) are expressed in quasilinear form with λ_n and $N_n(\infty)$ considered as parameters. The former is integrated to $s = 4$ with an assumed λ_n and $N_n''(0)$. Following this, (17) is integrated backwards from s^* (which can be selected from the above inequality once λ_n is chosen) to $s = 4$, with (18) as a boundary condition at s^*, and an assumed $N_n(\infty) = N_n(4)$. Continuity of N_n, N_n', N_n'' at $s = 4$ provides three conditions for the unknown λ_n, $N_n(\infty)$, $N_n''(0)$ and the constant associated with integration of (18). The fourth condition is found by integrating (14a) to $s = -10$ where (16) is applied.

The procedure was tested for accuracy by comparing the numerical results for the two smallest eigenvalues with the analytical forms in (15). The differences were of $O(10^{-3})$. The next three eigenvalues were found to be

$$\lambda_3 = 3.2743, \quad \lambda_4 = 4.6957, \quad \lambda_5 = 6.209 \tag{19}$$

Larger eigenvalues can, in principle, be computed. However the doubly infinite field and in particular the growth of s^* with λ_n implies increasingly large computation time.

4.2. *Asymptotic estimates of large eigenvalues*

Larger eigenvalues can be estimated by constructing an asymptotic theory based on the method described by Kemp [6]. In the present work some variation from Kemp's procedures are required in order to deal with the doubly infinite field of the mixing layer. The describing equation can be found by substituting the transformations

$$t = \int_0^s (F_0')^{\frac{1}{2}} ds, \quad v(t) = F_0'^{\frac{7}{4}} \exp\left[\left(\tfrac{1}{2}\right) \int_0^s F_0 ds\right](N/F_0')' \tag{20a, b}$$

into (14a). It follows that

$$v''(t) + (\mu - q(t))v = 0 \tag{21}$$

where

$$\mu = \lambda - (\quad), \quad q(t) = \frac{7}{16}\frac{F_0''^2}{F_0'^3} + \frac{F_0^2}{4F_0'} - \frac{5}{4}\frac{F_0''}{F_0'^2} - \frac{F_0 F_0''}{2F_0'^2} \tag{22}$$

It should be noted that the subscript n for λ and v have been dropped for convenience. The boundary conditions for (21), analogous to those in (14b) must be developed next. To this end we consider first the asymptotic behavior of $N'(s)$ for $s \to \pm\infty$. If the asymptotic formulas for the $F_0(s)$ function

$$F_0(s\to\infty) \sim s - k + \gamma(s-k)^{-2} \exp\left[-\left(\tfrac{1}{2}\right)(s-k)^2\right] + \dots, \tag{23a}$$

$$F_0(s\to -\infty) \sim -C_0 + a\,e^{C_0 s} + \dots, \quad k = 0.3739, \gamma = 0.198, a = 1.1502 \tag{23b}$$

are substituted into (14a), and the resulting equations are solved for $N'(s\to\infty)$ and $N(s\to-\infty)$, then it follows that

$$N'(s\to\infty) \sim A^+(s-k)^{(\lambda-1)} \exp\left[-\left(\tfrac{1}{2}\right)(s-k)^2\right] + \dots \tag{24a}$$

$$N'(s\to-\infty) \sim A^- + (B/C_0)\,e^{C_0 s} + \dots \tag{24b}$$

where A^+, A^- and B are integration constants.

One can combine (20a) and (23a) to show that for $s\to\infty$

$$t - \Delta \sim s - k, \qquad \Delta = \int_0^\infty ((F_0')^{\frac{1}{2}} - 1)\,ds + k\,. \tag{25a}$$

A similar manipulation with (20b) and (23b) implies that for $s\to-\infty$

$$t + \delta \sim 2(a/C_0)^{\frac{1}{2}} e^{(C_0 s/2)}, \quad \delta = -\int_0^{-\infty} (F_0')^{\frac{1}{2}} ds\,. \tag{25b}$$

The asymptotic behavior of $v(t)$ can now be obtained from (20b, 23, 24, 25). We find

$$v(t\to\infty) \sim (t-\Delta)^{(\mu-\frac{1}{2})} \exp\left[-\left(\tfrac{1}{4}\right)(t-\Delta)^2\right] \tag{26a}$$

$$v(t\to-\delta) \sim (t+\delta)^{\frac{1}{2}} + O([t+\delta]^{\frac{5}{2}}) \tag{26b}$$

corresponding to $s\to\pm\infty$ respectively.

Finally, conditions at the zero streamline, $s=0$, must be considered. Here

$$F_0(s\to0) \sim \alpha_1 s + (\alpha_2/2)s^2 - (\alpha_1\alpha_2/24)s^4 - (\alpha_2^2/120)s^5 + \dots \tag{27}$$

where $\alpha_1 = F_0'(0) = 0.58727012$, $\alpha_2 = F_0''(0) = 0.28242854$. Following procedures analogous to those outlined above, it is found that for $s\to0$

$$s \sim \alpha_1^{-\frac{1}{2}} t - (\alpha_2/4\alpha_1^2)t^2 + O(t^3) \tag{28}$$

$$v(t=0) = \alpha_1^{\frac{3}{4}} N'(0)\,. \tag{29}$$

A solution for $v(t;\mu)$ in (21), subject to (26) and (29), is sought in the limit $\mu\to\infty$. In particular, an explicit expression for the reduced eigenvalue $\mu = \lambda - \frac{1}{2}$ is to be obtained.

When $t \gg 1$, it can be shown from (22, 23a, 25a) that $q(t\to\infty) \sim (t-\Delta)^2/4$ so that (21) has the form

$$v''(t) + [\mu - ((t-\Delta)^2/4)]v = 0\,. \tag{30}$$

This equation has a turning point at $t \equiv t_0 = 2\mu^{\frac{1}{2}} + \Delta$ so that a WKB procedure is necessary. Following Kemp then it is observed that for $t > t_0$, the solution consists of exponentially increasing and decreasing functions. The latter has the asymptotic form given in (26a). The former is annihilated by the appropriate "matching" at $t=t_0$ of the $t>t_0$ solution with the $t<t_0$ solution. The latter is found to have the final form

$$v(t \gg 1) = (D/\mu^{\frac{1}{4}})\left(1 - \frac{(t-\Delta)^2}{4\mu}\right)^{-\frac{1}{4}} \cos(\chi-\theta), \quad t<t_0$$

$$\chi = \mu\left[\sin\left(\frac{t-\Delta}{2\mu^{\frac{1}{2}}}\right) - (\pi/2) + \frac{(t-\Delta)}{2\mu^{\frac{1}{2}}}\left(1 - \frac{(t-\Delta)^2}{4\mu}\right)^{\frac{1}{2}}\right] \tag{31}$$

$$\theta = (3\pi/4) + l\pi, \quad l=0, \pm 1, \pm 2, \ldots$$

Here the θ form arises from the required annihilation of the exponentially large terms in the $t>t_0$ solution [7]. The quantity D is an integration constant. If μ is thought of as a large parameter, then (31) is a solution which is formally valid for $t=O(\mu^{\frac{1}{2}})=O(\lambda^{\frac{1}{2}})$. When $t=o(\mu^{\frac{1}{2}})$ the asymptotic form of (31) is

$$v \sim (D/\mu^{\frac{1}{4}})\left[\cos\left(-\frac{\pi\mu}{2} + \mu^{\frac{1}{2}}(t-\Delta) - \theta + O(\mu^{-\frac{1}{2}})\right) + O(\mu^{-1})\right] \tag{32}$$

When $t \ll 1$, it can be shown from (22, 27) and (28) that $q \sim b_1 + b_2 t + O(t^2)$ where $b_1 = -7\alpha_2^2/16\alpha_1^3$, $b_2 = -[(3\alpha_2)/(4\alpha_1^{\frac{3}{2}})][1-(7\alpha_2^2/4\alpha_1^3)]$. Thus (21) has the special form

$$v'' + (\mu + b_1 + b_2 t)v = 0\,. \tag{33}$$

A general solution, in terms of Airy functions is

$$v(t) = C_1 A_i(-b_2^{\frac{2}{3}}[\mu + b_1 + b_2 t]) + C_2 B_i(-b_2^{\frac{2}{3}}[\mu + b_1 + b_2 t]) \tag{34}$$

where $C_{1,2}$ are integration constants. Since $|\mu + b_1 + b_2 t| \gg 1$, (34) can be rewritten in terms of the asymptotic expressions for $A_i(-x)$, $B_i(-x)$, $x \gg 1$ [8]. It follows that

$$v \sim (E/\mu^{\frac{1}{4}})\left[\cos\left(\frac{2\mu^{\frac{3}{2}}}{3b_2} + \mu^{\frac{1}{2}}\left[t + \frac{b_1}{b_2}\right] + (\pi/4) + F + O(\mu^{-\frac{1}{2}})\right) + O(\mu^{-1})\right] \tag{35}$$

where E and F are integration constants. A comparison of (32) and (35) can be used to show that

$$F = -\frac{2}{3}\frac{\mu^{\frac{3}{2}}}{b_2} - (\pi/2)\mu - \left(\frac{b_1}{b_2} + \Delta\right)\mu^{\frac{1}{2}} - \pi(1+l) \tag{36}$$

when $t \to -\delta$ it can be shown from (22), (23b) and (25b) that $q \sim -1/[4(t+\delta)^2]$ where δ is defined in (25b). Then (21) has the form

$$v'' + (\mu + [4(t+\delta)^2]^{-1})v = 0\,. \tag{37}$$

The solution of (37) which satisfies the boundary condition in (26b) is

$$v(t) \sim G(t+\delta)^{\frac{1}{2}} J_0(\mu^{\frac{1}{2}}[t+\delta]) \tag{38}$$

in which G is an integration constant and J_0 is the Bessel function. For $|\mu^{\frac{1}{2}}[t+\delta]| \gg 1$ the latter can be written in an asymptotic form. It follows that

$$v(t) = (G/\mu^{\frac{1}{4}})[\cos(\mu^{\frac{1}{2}}[t+\delta] - (\pi/4) + O(\mu^{-\frac{1}{2}})]\,.$$

If this is matched with (35), one finds that

$$F = -\frac{2}{3}\frac{\mu^{\frac{3}{2}}}{b_2} - \mu^{\frac{1}{2}}\frac{b_1}{b_2} + \mu^{\frac{1}{2}}\delta - (\pi/2) \tag{39}$$

A quadratic equation for $\mu^{\frac{1}{2}}$ can be found by equating (36) and (39);

$$\mu + (2/\pi)(\Delta+\delta)\mu^{\frac{1}{2}} + 2l + 1 = 0\,. \tag{40}$$

An explicit expression for $\mu^{\frac{1}{2}}$ can be found from (40). In order to insure real positive values of λ (see (22)), it must be asserted that $l=-n$, $n=1, 2, 3, \ldots$. This is in agreement with the definition of l in (31). Then it follows that

$$\lambda_n = 2n - \tfrac{1}{2} + (\omega^2/2) - \omega(2n - 1 + (\omega^2/4))^{\frac{1}{2}} + O(n^{-\frac{1}{2}})$$
$$\omega = (2/\pi)(\Delta + \delta) . \tag{41}$$

Numerical evaluation of the formulas in (25) show that $\Delta = 0.173$, $\delta = 2.09$. The asymptotic values of λ_n for the first five eigenvalues are compared with the numerically obtained values in Table 1. The agreement is favorable.

TABLE 1

n	λ_{exact}	$\lambda_{\text{asymp.}}$
1	1	.7624
2	2	1.8358
3	3.2743	3.1541
4	4.6957	4.5884
5	6.209	6.0942

5. Wall-layer corrections

The form of the first wall-layer correction can be ascertained from the streamfunction matching condition, $f(x, \eta \to \eta^*) \sim F(x, s \to -\infty)$. This has the explicit form

$$f(x, \eta^*) + f'(x, \eta^*)(\eta - \eta^*) + \ldots \sim C_0 + \sum_{n=1}^{\infty} a_n x^{-\lambda_n/2} N_n(-\infty) . \tag{42}$$

Here $(\eta - \eta^*)$ can be expressed in terms of $y_0(x; \text{Re})$ by combining the transformations in (2) and (6), the definition of $y_0(x; \text{Re})$ in (9). It follows that

$$\eta - \eta^* = (2x)^{-\frac{2}{3}} \sum_{n=1}^{\infty} g_n(x) + O(\text{Re}^{-\frac{1}{6}}) \tag{43}$$

where $g_1(x) = o(x^{\frac{2}{3}})$. Then if (4a), the value $f_0(\eta^*) = -C_0$ and (43) are substituted into (42), it follows that

$$f_1(x, \eta^*) + f_0'(\eta^*)(2x)^{-\frac{2}{3}} g_1(x) = \sum_{n=1}^{\infty} a_n x^{-\lambda_n/2} N_n(-\infty) . \tag{44}$$

Eq. (44) implies that the expansions for $f_1(x, \eta)$ and $g_1(x)$ have the form

$$f_1 = \sum_{n=1}^{\infty} \alpha_n x^{-\lambda_n/2} G_n(\eta) \tag{45a}$$

$$g_1 = \sum_{n=1}^{\infty} \delta_n (2x)^{\frac{2}{3}} x^{-\lambda_n/2} \tag{45b}$$

where α_n, δ_n are constants to be found. The matching condition for $G_n(\eta)$ can then be extracted from (44) and (45)

$$\alpha_n G_n(\eta^*) = a_n N_n(-\infty) - f_0'(\eta^*) \delta_n . \tag{46}$$

The dividing streamline equation, (9), can now be written as

$$y_0(x, \text{Re}) = \text{Re}^{-\frac{1}{3}} \left\{ (2x)^{\frac{2}{3}} \left[\eta^* + \sum_{n=1}^{\infty} \delta_n x^{-\lambda_n/2} \right] + O(g_2) + O(\text{Re}^{-\frac{1}{6}}) \right\} . \tag{47}$$

Since (47) represents the shape of the effective displacement body, it follows from interaction considerations that the pressure field $p = \text{Re}^{-\frac{1}{3}} \bar{p}$ is proportional to $y_0'(x; \text{Re})$ or $p' \sim y_0''(x; \text{Re})$. Thus (4b) can be written as

$$\beta(x)=\beta_0+\sum_{n=1}^{\infty}\gamma_n x^{-\lambda_n/2} \tag{48}$$

where γ_n remains to be determined.

The systems describing the G_n functions are derived from (3), (4), (45a) and (48). It follows that for $n\geqq 1$

$$\begin{aligned} &f_0 G_n''+f_0'((^2/_3)+\lambda_n)G_n'+f_0''(1-\lambda_n)G_n=-\gamma_n/\alpha_n \\ &G_n(0)=G_n'(0)=0\,. \end{aligned} \tag{49}$$

An analytical solution for (49) is possible because one exact closed form homogeneous solution, f_0', exists. Hence the full solution has the form

$$G_n=(\gamma_n/\alpha_n)H_n(\eta,\lambda_n) \tag{50}$$

where

$$H_n(\eta;\lambda_n)=-\int_0^{\eta}([\phi_1(t)\phi_2(\eta)-\phi_1(\eta)\phi_2(t)]/f_0(t)W(t))dt$$

$$\phi_1(t)=f_0'(t)\,,$$

$$\phi_2(t)=f_0'(t)\int_0^t \exp\left[\int_0^s\left(\frac{2\beta_0-\lambda_n f_0'^2(\sigma)}{f_0(\sigma)f_0'(\sigma)}\right)d\sigma\right]ds$$

$$W(t)=\phi_1(t)\phi_2'(t)-\phi_1'(t)\phi_2(t)\,.$$

It is noted that since $\alpha_n G_n=\gamma_n H_n(\eta,\lambda_n)$, then

$$f_1=\sum_{n=1}^{\infty}\gamma_n H_n(\eta,\lambda_n)x^{-\lambda_n/2}\,. \tag{51}$$

Hence the expansion constants in (48) and (51) are the same. Furthermore from (46) and (50) we obtain

$$\gamma_n H_n(\eta^*,\lambda_n)=a_n N_n(-\infty)-f_0'(\eta^*)\delta_n \tag{52}$$

Since $H_n(\eta^*,\lambda_n)$, $N_n(-\infty)$ and $f_0'(\eta^*)$ are known, and the a_n are Fourier coefficients which depend on the initial profile, then (52) provides one relation for the two unknowns γ_n and δ_n. A second relation follows from an explicit calculation of the pressure interaction with the external field. Eqs. (10), (11), and (45b) can be combined to produce an expression for the corrections to the similarity pressure distribution

$$\begin{aligned} &\bar{p}-(3\beta_0/2)(2x)^{-\frac{1}{3}}=\frac{2^{\frac{2}{3}}}{\pi}\int_{x_i}^{\infty}\sum_{n=1}^{\infty}\left(\frac{2}{3}-\frac{\lambda_n}{2}\right)\delta_n\frac{\xi^{-\mu_n}}{(x-\xi)}d\xi \\ &\mu_n=(\lambda_n/2)+(\tfrac{1}{3})\,. \end{aligned} \tag{53}$$

The integral in (53) can be evaluated by dividing the interval into the ranges $x_i\leqq\xi\leqq x-\varepsilon$, $x+\varepsilon\leqq\xi\leqq\infty$ and rewriting the integrands in the appropriate uniformly convergent series form. Then for the limit $\varepsilon\to 0$, $x\to\infty$ (53) becomes

$$\bar{p}-(3\beta_0/2)(2x)^{-\frac{1}{3}}=\frac{2^{\frac{2}{3}}}{\pi}\sum_{n=1}^{\infty}\left[(1-\mu_n)(2\mu_n-1)\delta_n\sum_{m=0}^{\infty}(m+1-\mu_n)^{-1}(m+\mu_n)^{-1}\right]x^{-\mu_n} \tag{54}$$

Then it follows from the definition of $\beta(x)$ in (2), and (4b), (48) and the derivative of (54) that

$$\gamma_n=(4/\pi)\mu_n(1-\mu_n)(2\mu_n-1)\delta_n\sum_{m=0}^{\infty}[(m+1-\mu_n)(m+\mu_n)]^{-1}\,. \tag{55}$$

The infinite series in (55) has strong convergence properties. As $m\to\infty$ the terms approach m^{-2}. Thus from (52) and (55) explicit numerical values of γ_n, δ_n can be found.

6. Summary and conclusions

The wall and shear-layer solutions can now be written as

$$(2x)^{-\frac{1}{2}}\bar{\psi} = f(x, \eta) = f_0(\eta) + \sum_{n=1}^{\infty} \gamma_n H_n(\eta) x^{-\lambda_n/2} \tag{56a}$$

$$(2x)^{-\frac{1}{2}}\tilde{\psi} = F(x, s) = F_0(s) + \sum_{n=1}^{\infty} a_n N_n(s) x^{-\lambda_n/2} \tag{56b}$$

$$y_0(x; \mathrm{Re}) = \mathrm{Re}^{-\frac{1}{3}}(2x)^{\frac{2}{3}}\left[\eta^* + \sum_{n=1}^{\infty} \delta_n x^{-\lambda_n/2}\right] \tag{56c}$$

$$\bar{p}'(x) = -(2x)^{-\frac{4}{3}}\left[\beta_0 + \sum_{n=1}^{\infty} \gamma_n x^{-\lambda_n/2}\right]. \tag{56d}$$

Here γ_n, δ_n are known functions of the Fourier coefficients a_n. Thus with given initial data at x_i in the shear layer (see 1f) each of the coefficients can be calculated. Then γ_n, δ_n can be explicitly evaluated from (52) and (55). It is to be noted that the coefficients in the wall-layer expansions are essentially prescribed by the shear-layer eigenfunction behavior. Hence the initial value of the velocity profile in the wall-layer (see 1e) can, in a sense, be calculated from (56a). This is interpreted to mean that the wall-layer initial data must be compatible with that in the shear layer. This is not surprising because the development of the two layer upstream of x_i is interrelated by the interaction process.

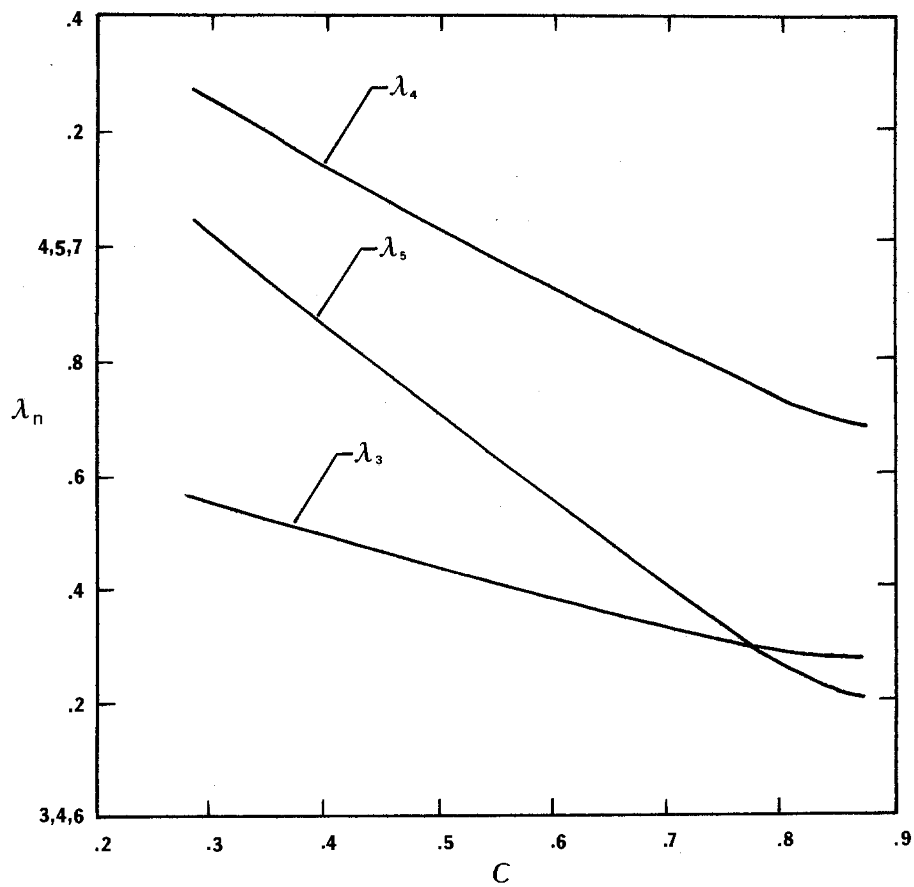

Figure 1. Eigenvalues *versus* injection parameter C. The origin for λ_n, (3, 4, 6) corresponds to $n = 3, 4, 5$ respectively.

The results in (56) show that the eigenfunctions generated in the shear layer cause disturbances in the wall layer and external flow through the mechanism of the interaction phenomena. The asymptotic rate of decay of the initial profiles to the equilibrium similarity solutions is like $(x)^{-\frac{1}{2}}$ for $x\to 0$ because the smallest eigenvalue $\lambda_0 = 1$. This may be compared with the classical result for flat plate boundary-layer flow [9] where the decay rate is $O(x^{-1})$ corresponding to $\lambda_0 = 2$. Evidently free shear layers are less spatially-stable than boundary layers of a related type.

Krishnamurthy and Williams [10] have considered the boundary-layer eigenvalue problem arising from a linearization about the injected Blasius profile [11] for values of the wall blowing parameter $-f(0) = C < C_0$. They computed numerical values for the first several eigenvalues as a function of C. In Fig. 1 these results are shown for $n = 3, 4, 5$. The largest injection rate considered by Krishnamurthy and Williams is $C = 0.81317$. The values at $C = 0.87574\ldots = C_0$ are those found in Section 3 of the present work. They represent the natural extension of the boundary-layer results to the critical blowoff configuration. Of course larger values of injection cannot be considered within the framework of boundary-layer theory. However, the present theory shows that for a flat plate with a wall injection rate larger than C_0, the eigenvalues are those of the free-shear layer. Hence beyond $C = C_0$ the eigenvalues remain constant.

Appendix A

Sturm–Liouville system

The transformation $H_n = (N_n/F_0')'$ applied to (14a) leads to

$$L(H_n) \equiv \left(\frac{F_n'^3}{F_0''} H_0'\right)' - F_0 F_0'^2 H_n = -\lambda_n \cdot \frac{F_0'^4}{F_0''} H \,. \tag{A1}$$

The boundary conditions on $H_n(s)$ are

$$H_n(s\to\infty) \sim (s-k)^{\lambda-1} \exp\left[-(\tfrac{1}{2})(s-k)^2\right] \tag{A2}$$

$$H_n(0) = N'(0)/\alpha \,, \tag{A3}$$

$$H_n(s\to -\infty) \sim A_n e^{-C_0 s}, \quad A_n = \text{constant} \tag{A4}$$

where use has been made of (23) and (24). In order to show that $L(H_n)$ is a self adjoint operator, let H_n, H_m be two solutions of (A1) satisfying A(2–4). Then

$$\begin{aligned}\int_{-\infty}^{+\infty} [H_n L(H_m) - H_m L(H_n)]\, ds &= \lim_{R_1\to\infty, R_2\to-\infty} \left[\frac{F_0'^3}{F_0''}(H_n H_m' - H_m H_n')\right]_{R_2}^{R_1} \\ &= \lim_{R_2\to-\infty} \frac{a^3 C_0^3 e^{3C_0 s}}{a C_0^2 e^{C_0 s}} [A_n A_m C_0 - A_n A_m C_0] e^{-2C_0 s} \\ &= 0\end{aligned} \tag{A5}$$

where use has been made of (A4) and (23b). It follows immediately that the eigenvalues are real and that the eigenfunctions are orthogonal on $(-\infty, \infty)$ with respect to the weight function $(F_0')^4/F_0''$.

With regard to the positiveness of the eigenvalues, use of (A1) and (A5) leads to

$$\lambda_n \int_{s_1}^{\infty} (F_0'^4/F_0'') H_n^2\, ds = \int_{s_1}^{\infty} F_0 F_0'^2 H_n^2\, ds + \int_{s_1}^{\infty} (F_0'^3/F_0'') H_n'^2\, ds \tag{A6}$$

where $s_1 > 0$ is such that $H(s_1) = 0$. That there is such an s_1 is a result of the trigonometric nature of solution (31) and definition (20b) of $v(t)$. Now all integrals appearing in (A6) are positive which leads to $\lambda_n > 0$.

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